Stable Matching with Adaptive Priorities

FEDERICO BOBBIO, CERC, CIRRELT and DIRO, Université de Montréal, Canada
MARGARIDA CARVALHO, CIRRELT and DIRO, Université de Montréal, Canada
IGNACIO RIOS, School of Management, University of Texas at Dallas, U.S.A.
ALFREDO TORRICO, CDSES, Cornell University, U.S.A.

We study the problem of finding a stable matching under adaptive priorities, whereby the clearinghouse prioritizes some agents based on the allocation of others, and we use school choice as a motivating example. To accomplish this, we introduce a stylized model of a two-sided matching market with siblings' priorities. We argue that the standard notion of stability does not apply in the presence of adaptive priorities. To address this, and motivated by practice, we define several assumptions on families' preferences and siblings' priorities, introduce different notions of stability under adaptive priorities, and show that a stable matching under these settings may not exist. On the positive side, we show that such matchings exist if families strictly prefer their members to remain together in two important settings: (i) when families are of size at most two, and (ii) when there is a single grade level. In addition, we devise a mechanism to find such stable assignments in polynomial time. Finally, we show that the problem of finding the stable matching under adaptive priorities of maximum cardinality is NP-Hard.

Additional Key Words and Phrases: stable matching, school choice, families, adaptive priorities

ACM Reference Format:
Federico Bobbio, Margarida Carvalho, Ignacio Rios, and Alfredo Torrico. 2023. Stable Matching with Adaptive Priorities. 1, 1 (September 2023), 18 pages. https://doi.org/XXXXXXX.XXXXXXX

1 INTRODUCTION

The theory of two-sided many-to-one matching markets, introduced by Gale and Shapley [10], provides a framework for solving many large-scale real-life assignment problems. Examples include entry-level labor markets for doctors and teachers, education markets from daycare, school choice and college admissions, and other applications such as refugee resettlement. A common feature in many of these markets is the use of mechanisms that find a stable assignment, as this guarantees that no coalition of agents has incentives to circumvent the match.

In many of these markets, the clearinghouse may be interested in finding a stable allocation, while individual agents may care about their assignment and that of other agents. For instance, in the hospital-resident problem, couples jointly participate and must coordinate to find two positions that complement each other. In school choice, students may prefer to be assigned with their siblings or neighbors. In refugee resettlement, agencies may prioritize allocating families with similar backgrounds (e.g., from the same region or speaking the same language) to the same cities.

Authors' addresses: Federico Bobbio, CERC, CIRRELT and DIRO, Université de Montréal, Montréal, Canada, federico.bobbio@umontreal.ca; Margarida Carvalho, CIRRELT and DIRO, Université de Montréal, Montréal, Canada, carvalho@iro.umontreal.ca; Ignacio Rios, School of Management, University of Texas at Dallas , Dallas, U.S.A., ignacio.riosuribe@utdallas.edu; Alfredo Torrico, CDSES, Cornell University, New York, U.S.A., alfredo.torrico@cornell.edu.

Manuscript submitted to ACM

One approach to accommodate these joint preferences is to provide priorities contingent on the assignment. For instance, many school choice systems (including NYC, New Haven, Denver, Chile, etc.) consider *sibling priorities*, by which students get prioritized in schools where they have a sibling currently assigned or enrolled. Similarly, in refugee resettlement, families may get higher priority in localities where they have relatives based on *family reunification*. However, most clearinghouses assume that priorities are fixed and known before the assignment process and, thus, cannot accommodate settings in which priorities depend on the current assignment. Similarly, most definitions of stability and justified-envy assume that priorities are fixed and known, and thus, also fail to capture the aforementioned setting.

In this paper, we study the problem of finding a stable matching under *adaptive priorities*, i.e., when priorities are updated based on the current assignment, and we use school choice with siblings as a motivating example. To accomplish this, we first introduce a stylized model where students belong to (potentially different) grade levels and may have siblings applying to the system (potentially in different levels). On the one hand, each family reports preferences over the assignment of their members while, on the other hand, schools prioritize students with siblings already enrolled or currently assigned, and break ties among students in the same priority group (with or without siblings) using a random tie-breaker.

Motivated by the Chilean school choice system, we distinguish two types of sibling priority: (i) static, whereby students who have a sibling currently enrolled but not participating in the admission process get prioritized; and (ii) adaptive, whereby students with a sibling who is also participating in the admission process and is currently assigned get prioritized. Notably, adaptive priorities depend on the current assignment and, thus, must be granted and updated simultaneously while solving the allocation task. This simultaneity introduces a series of challenges, and the standard notion of *justified envy* no longer applies.

To overcome these challenges, we start by simplifying the space of families' preferences and introducing several assumptions that limit how adaptive priorities work. Specifically, we present the concepts of absolute/partial and dependent/independent justified envy to break ties across and within priority groups. Based on these definitions, we introduce several notions of stability and show that a stable matching may not exist. Nevertheless, we show that a stable matching under adaptive priorities exists if families strictly prefer that their siblings are assigned together and either (i) families have at most two members participating in the admission process or (ii) there is a single grade level. Moreover, we introduce a new family of mechanisms that find such a stable matching. Finally, we discuss other properties of the mechanism, and we show that finding a student-optimal or a maximum cardinality stable matching under adaptive priorities is NP-hard.

Our work contributes to the literature in several ways. To the best of our knowledge, this is the first work to formalize different types of siblings' priorities and also the first to introduce the idea of adaptive priorities. Consequently, we introduce a novel notion of stability under adaptive priorities, where priorities are contingent on the matching. We also provide the first complexity results for a stable matching problem with complementarities. Although we focus on school choice as a motivating example, our results and insights may deem helpful in the design of matching mechanisms where priorities depend on the assignment of others, such as in daycare assignments, college admissions, refugee resettlement, among others.

1.1 Organization

The remainder of this paper is organized as follows. In Section 2, we discuss the relevant literature. In Section 3, we introduce our model. In Section 4, we study the existence of a stable assignment under adaptive priorities. In Section 5, we study the complexity of finding a maximum cardinality stable matching under adaptive priorities. Finally, in Section 6 we conclude.

2 LITERATURE

Our paper is related to several strands of the literature.

Matching with families. A recent strand of the literature has extended the classic school choice model [2] to incorporate families. Dur et al. [8] consider a setting where siblings report the same preferences, and assignments are feasible if and only if all family members are assigned to the same school (or all of them are unassigned). The authors argue that justified envy is not an adequate criterion for the problem. Thus, they propose a new solution concept (suitability), show that a suitable matching always exists, and introduce a new family of strategy-proof mechanisms that finds a suitable matching. Correa et al. [6] also consider a model with siblings applying to potentially different grades, but assume that each sibling submits their own (potentially different) preference list. In addition, the authors assume that the clearinghouse aims to prioritize the joint assignment of siblings, but they model it as a soft requirement, i.e., an assignment may be feasible even if siblings are not assigned to the same school. To prioritize the joint assignment of siblings, Correa et al. [6] introduce (i) the use of lotteries at the family level; (ii) a heuristic that processes grades sequentially in decreasing order, updating priorities in each step to capture siblings' priorities that result from the assignment of higher grades; and (iii) the option for families to report that they prefer their siblings to be assigned to the same school rather than following their individual reported preferences. This last feature, called *family application*, prioritizes the joint assignment of siblings by updating the preferences of younger siblings by adding the school of assignment of their older siblings. The authors show that all these features significantly increase the probability that families get assigned together.

Matching with couples. Our paper is also related to the matching with couples literature, which is commonly motivated by labor markets such as the matching for medical residents. In this setting, couples wish to be matched in the same hospital and hence, they report a joint preference list of pairs of hospitals. For an extension of the stability concept with couples, Roth [21] shows that a stable matching may not exist if couples participate. To overcome this limitation, Klaus and Klijn [12], Klaus et al. [14] introduce the property of weak responsive preferences and show that this guarantees the existence of a stable assignment. Kojima et al. [16] provide conditions under which a stable matching exists with high probability in large markets, and introduce an algorithm that finds a stable matching with high probability which is approximately strategy-proof. Ashlagi et al. [3] find a similar result, as they show that a stable matching exists with high probability if the number of couples grows slower than the size of the market. However, the authors also show that a stable matching may not exist if the number of couples grows linearly. Finally, Nguyen and Vohra [19] show that the existence of a stable matching is guaranteed if the capacity of the market is expanded by at most 9 spots.

Matching with complementarities. Beyond families and couples, the matching literature has studied other settings with complementarities. For instance, Ashlagi and Shi [4] shows that using correlated lotteries can increase community cohesion by increasing the probability of neighbors being assigned to the same schools. Dur and Wiseman [9] also study the matching problem with neighbors and show that a stable matching may not exist if students have preferences over joint assignment with their neighbors. Moreover, the authors show that the student-proposing deferred acceptance algorithm is not strategy-proof and propose a new algorithm to address these issues. Kamada and Kojima [11] study matching markets where the clearinghouse cares about the composition of the match and thus imposes distributional constraints. The authors show that existing mechanisms suffer from inefficiency and instability and propose a mechanism that addresses these issues while respecting the distributional constraints. Nguyen and Vohra [20] also study the problem with distributional concerns but consider these constraints as soft bounds and provide ex-post guarantees on how close the constraints are satisfied while preserving stability. Nguyen et al. [18] introduce a new model of many-to-one matching where agents with multi-unit demand maximize a cardinal linear objective subject to multidimensional knapsack constraints, capturing settings such as refugee resettlement, day-care matching, and school choice/college admissions with diversity concerns. The authors show that a pairwise stable matching may not exist and provide a new algorithm that finds a group-stable matching that approximately satisfies all the multidimensional knapsack constraints. Another example in which agents care about other agents' assignment is the affiliate stable matching problem, where, for instance, a college is not only interested in hiring good academic candidates, but also wishes that its graduates find good jobs [7, 15].

3 MODEL

In this section, we introduce a two-sided matching market model that includes a priority system. To facilitate the exposition, we use school choice with sibling priorities as a concrete application of the model.

Let $\mathcal{S}$ be a finite set of students and $\mathcal{F} \subseteq 2^{\mathcal{S}}$ be a partition of $\mathcal{S}$ where $f \in \mathcal{F}$ is called a *family* and its size is denoted as $|f|$. For $f \in \mathcal{F}$ with $|f| \geq 2$, we say that students s and s' are *siblings* if $s, s' \in f$. If $f \in \mathcal{F}$ is such that $f = \{s\}$, then we say that s has no siblings. With a slight abuse of notation, we define function $f : \mathcal{S} \to \mathcal{F}$ to map a student into their specific family, i.e., each student $s \in \mathcal{S}$ belongs to family $f(s) \in \mathcal{F}$. Note that students s and s' are siblings if $f(s) = f(s')$ and a student s has no siblings when $f(s) = \{s\}$.

Let C be a finite set of schools and $\mathcal{G}$ be the set of grade levels. With a slight abuse of notation, we define a function $g : \mathcal{S} \to \mathcal{G}$ that maps a student $s \in \mathcal{S}$ into the grade level $g(s)$ to which they are applying to. We denote by $\mathcal{S}^g \subseteq \mathcal{S}$ the set of students applying to grade level $g \in \mathcal{G}$. We assume that each school $c \in C$ offers $q_c^g \in \mathbb{Z}_+$ seats on grade level $g \in \mathcal{G}$, where $q_c^g = 0$ means that school c does not offer grade g.

Let $\mathcal{E} \subseteq \mathcal{S} \times C \cup \{\emptyset\}$ be the set of feasible pairs, i.e., $(s, c) \in \mathcal{E}$ implies that student s and school c deem each other acceptable and $q_c^{g(s)} > 0$; $\emptyset$ represents being unassigned. A matching is an assignment $\mu \subseteq \mathcal{E}$ such that (i) each student is assigned to at most one school in C, and (ii) each school is assigned at most its capacity in each grade level. Formally, for $\mu \subseteq \mathcal{E}$, let $\mu(s) \in C \cup \{\emptyset\}$ be the school that student s was assigned to, $\mu(f) \subseteq C$ be the subset of schools where the students of family f were assigned to, i.e., $\mu(f) = \{\mu(s) : s \in f\}$, and $\mu(c) \subseteq \mathcal{S}$ be the set of students assigned to school c. Given a grade g, we denote by $\mu^g(c)$ the set of students assigned to school c at grade g. Then, a matching satisfies that (i) $\mu(s) \in C \cup \{\emptyset\}$ for all students $s \in \mathcal{S}$ and (ii) $|\mu^g(c)| \leq q_c^g$ for all schools $c \in C$ and grade levels $g \in \mathcal{G}$.[1]

Each family $f = \{s_1, \ldots, s_\ell\} \in \mathcal{F}$ has a strict preference order $\succ_f$ over tuples in $(C \cup \{\emptyset\})^\ell$, which means that $(c_1, \ldots, c_\ell) \succ_f (c'_1, \ldots, c'_\ell)$ implies that family f prefers that its members $s_1, \ldots, s_\ell$ go to schools $c_1, \ldots, c_\ell$ over $c'_1, \ldots, c'_\ell$, respectively. On the other hand, each school $c \in C$ has a strict preference order $\succ_c$ over feasible subsets of $\mathcal{S}$ which means that for subsets $S, S' \subseteq \mathcal{S}$ that satisfy grade level capacities, $S \succ_c S'$ denotes that school c prefers students in S over students in S'.

As Roth [23] discusses, a desired property of any matching is stability, i.e., that there is no group of agents that prefer to circumvent their current match and be matched to each other. Given a matching $\mu \subseteq \mathcal{E}$, we say that student s has *justified envy* towards another student s' assigned to school c if (i) $g(s) = g(s')$, (ii) $(c, \mu(f \setminus \{s\})) \succ_f \mu(f)$, and (iii)

[1] Notice that the model captures other single-level applications such as refugee resettlement, college admissions and the hospital-resident problem.

$(\mu(c) \cup \{s\}) \setminus \{s'\} \succ_c \mu(c)$. In words, the first condition states that both students belong to the same grade level; the second condition implies that the family prefers that $s \in f$ is assigned to c rather than $\mu(s)$, given the assignment of their siblings; and the third condition states that school c prefers the set of student that replaces s' with s. In addition, we say that a matching μ is *non-wasteful* if there is no student $s \in \mathcal{S}$ and school c such that $(c, \mu(f \setminus \{s\})) \succ_f \mu(f)$ and $|\{s' \in \mu(c) \ : \ g(s') = g(s)\}| < q_c^g$. Finally, we say that a matching is *stable* if no student has *justified envy* and it is *non-wasteful*.

To account for *sibling priorities*, we aim to reshape the space of preferences of the schools. Sibling priorities can happen in two forms:

(1) *Static priority*: A family $f \in \mathcal{F}$ has static priority in school c if one or more students in f are applying to c and have a sibling who is currently enrolled in c and is not participating in the admission process.[2] Therefore, school c prefers each student in f over students in $\mathcal{S}$ with no sibling priority.

(2) *Adaptive priority*: A family $f \in \mathcal{F}$ has adaptive priority in school c if two or more students in f are simultaneously applying to c. Therefore, school c prefers those students in f over students in $\mathcal{S}$ with no sibling priority. This type of priority is called adaptive because students get prioritized only if another sibling is assigned to the school, i.e., priorities adapt to the current matching.

Throughout the paper, we often shorten sibling priority as priority. Under static (resp. adaptive) priorities, we say that student s *provides* sibling priority in school c if s is currently enrolled (resp. assigned) in c, and we say that the siblings of s *receive* sibling priority in school c. Note that a student may receive static and adaptive priority in different schools or both types of priority in the same one. For instance, suppose that a family $f = \{s, s'\}$ is applying to schools c and c', and that s and s' have a sibling $s'' \notin \mathcal{S}$ currently enrolled in c and not applying to the system. If s, who receives static priority from s'' in school c, gets assigned to school c' in the current matching,[3] then s' would receive static priority in c and adaptive priority in c'. In contrast, if s gets assigned to c, then s' receives both static and adaptive priority in c. Therefore, we assume that static priority overrules adaptive priority, i.e., a student with potentially both priorities in a given school can only benefit from the static priority.[4] In other words, students are not additionally prioritized if they have siblings enrolled and also siblings currently matched. We borrow this assumption from practice, as in certain school districts (e.g., in Chile), the clearinghouse prefers to assign students with static priority because their probability of enrollment is higher compared to students without siblings currently enrolled.

Given the above, in practice, these priorities define three disjoint groups of applicants in each school: (i) students with static priority, (ii) students with adaptive priority, and (iii) students with no priority. Within each group, all students are equally preferred by the school, and thus the clearinghouse breaks ties using a random tie-breaker. Note that if there are only students with no priority and families with static priorities, then the random tie-breaker defines a strict order over the whole set students $\mathcal{S}$ in each school, as the group with siblings will be always prioritized over the group with no siblings. Thus, in this case, for any school $c \in \mathcal{C}$, $\succ_c$ would be as if no student had siblings, but with the group of students with sibling priority placed first in the list and then the rest.[5] This implies the following immediate corollary.

Corollary 3.1 ([10]). *If there are no students who can receive adaptive priority, then a stable matching exists.*

[2] This means that this sibling is not part of the input $\mathcal{S}$.

[3] This could happen if the family prefers s to be assigned in school c', or it could happen if school c is over-demanded and all the seats are filled with students with static sibling priority.

[4] In the example above, s' would only have static priority.

[5] In other words, the static priority and the random tie-breaking rule define a unique set ordering $\succ_c$ which translates in a linear preference order.

Given this positive result, we focus our attention on adaptive priorities where the existence may not be guaranteed. Henceforth, we consider the following assumption.

Assumption 3.1. *No student has static priority in any school. Thus, in each school, the set of students are composed by two disjoint groups of applicants: (i) students with (adaptive) sibling priority, and (ii) students with no priority.*

In the remainder of the paper, we use sibling priority and adaptive priority interchangeably. In addition, we assume that schools break ties within each group with a random tie-breaker and we denote by $p_{s,c} \in \mathbb{R}_+$ the value of the random tie-breaker of student s for school c. As opposed to static priorities, the combination of adaptive priorities and random tie-breakers do not define a unique order among any two pair of students for each school, as this pair may change from one priority class to the other depending on the current match of their siblings. In fact, the existence of a stable matching is not guaranteed as shown in [6] (see their Proposition 1).

The main challenge with adaptive priorities is the dependency on the *current matching*. Specifically, consider a family $f = \{s, s'\}$ and a matching mechanism that, at some step, matches student s to school c and student s' to some school $c' \in C \cup \{\emptyset\} \setminus \{c\}$ such that $(c, c) \succ_f (c, c')$. Then, s' has adaptive priority in c, and the mechanism would assign s' to c in grade level $g(s')$, potentially displacing another student $s'' \notin f$ without priority applying to the same grade $g(s')$. Given that multiple families are simultaneously applying to different schools and grade levels, a stable matching may not exist as we previously mentioned. To address this simultaneity challenge, school districts have either (i) defined an order to process grades, and the clearinghouse updates adaptive priorities before moving to the next grade [6]; or (ii) do not consider adaptive priorities. As we discuss in Appendix A, different processing order of grade levels lead to different outcomes.

The design of adaptive sibling priorities opens four immediate important questions. First, what is an appropriate notion of stability to capture adaptive priorities? Second, under which assumptions can we guarantee the existence of a stable matching? Third, if such assumptions exist, can we find a stable matching under adaptive priorities efficiently? And finally, what are the properties of these stable matchings? Our goal in the next section is to simplify the space of preferences and formalize how siblings' priorities affect schools' ordering of students, so as to properly define a new notion of stability that considers adaptive priorities.

3.1 Simplifying the space of preferences and priorities

The definition of *justified envy* in the previous section assumes that schools have preferences over sets of students, and that families have joint preferences over tuples of schools. However, in most clearinghouses, preferences are not as complex. In practice, they involve students declaring linear preferences over schools, and schools' linear preferences are determined by a combination of random tie-breakers and priority groups. For this reason, in the remainder of the paper, we assume a simplified structure of preferences, as formalized in the following Assumption 3.2.

Assumption 3.2. *We assume the following structure for preferences and tie-breaking rules:*

(1) On the students' side, we assume that each family reports a strict preference order over $C \cup \{\emptyset\}$ and that each family member s follows the same preference order as their family among the schools that offer grade $g(s)$.

(2) On the schools' side, we assume that every school has sibling priority and uses a random tie-breaker to break ties between students in the same applicant group (i.e., students with or without sibling priority).

Although Assumption 3.2 simplifies the reporting of preferences, the sibling priority needs some limitations to ensure the fairness of the assignment, as the following example illustrates.

Example 3.2. Consider an instance with a single level, a set of students $\mathcal{S} = \{a_1, a_2, a_3, s_1, s_2, s'_1, s'_2\}$ where $f = \{s_1, s_2\}$ and $f' = \{s'_1, s'_2\}$ are siblings, and a single school c with capacity 4. Moreover, suppose the random-tie breakers of school c are $p_{a_1,c} > p_{a_2,c} > p_{a_3,c} > p_{s_1,c} > p_{s_2,c} > p_{s'_1,c} > p_{s'_2,c}$. Then, one possible matching is $\mu = \{(a_1, c), (a_2, c), (a_3, c), (s_1, c), (s_2, \emptyset), (s'_1, \emptyset), (s'_2, \emptyset)\}$. However, the alternative matchings

$$\mu' = \{(a_1, \emptyset), (a_2, \emptyset), (a_3, \emptyset), (s_1, c), (s_2, c), (s'_1, c), (s'_2, c)\}$$

and

$$\mu'' = \{(a_1, c), (a_2, c), (a_3, \emptyset), (s_1, c), (s_2, c), (s'_1, \emptyset), (s'_2, \emptyset)\}$$

are also feasible in terms of capacity, but depending on how siblings are prioritized over students with no siblings, one would be more desirable than the other. □

Note that in Example 3.2, matching μ' is not desirable, since neither s'_1 nor s'_2 would be admitted in school c without adaptive priority. This differs from the case of family f, because there is a matching μ that only accounts for random-tie breakers and no sibling priority in which s_1 is matched to c and, consequently, could potentially provide adaptive priority to s_2. To rule out this issue, we restrict our attention to matchings that satisfy the following assumption.

Assumption 3.3. *A student cannot simultaneously provide and receive sibling priority in a given school.*

Note that the assignment μ'' in Example 3.2 satisfies Assumption 3.3 and thus is a feasible matching with sibling priority. On the other hand, μ' does not satisfy this assumption, because neither s'_1 nor s'_2 would be assigned in μ' if the other is not part of $\mathcal{S}$.

Given Assumption 3.3, the key question is which matchings the clearinghouse prefers. As we saw in Example 3.2, sibling priority in μ'' leads to s_2 displacing other students previously assigned in c. Before focusing on which matchings are preferred, we ask ourselves the following: Is a student with sibling priority "allowed" to displace any other student without priority? This question leads us to define two notions of priorities: (1) *Absolute priority* in which a prioritized student s in school c can displace any other student with no priority, regardless of their random tie-breaker; and (2) *partial priority* in which a prioritized student s in school c can displace only certain students with no priority.[6]

Both notions of sibling priority have implications in terms of justified-envy and, consequently, for the stability of the matching. Therefore, we formalize the concepts of *absolute* and *partial* justified-envy in Definition 3.3. For this, let $P_\mu(s, c) = \max_{a \in f(s) \setminus \{s\}} \{p_{a,c} \ : \ \mu(a) = c, \ a \succ_c s\}$ be the function that returns the highest random tie-breaker among the siblings of student s currently assigned to c. If the student s does not have a sibling currently assigned to c, then we define $P_\mu(s, c) = p_{s,c}$.

Definition 3.3 (Absolute and partial justified-envy). Consider a matching $\mu \subseteq \mathcal{E}$.

- A student with sibling priority s has *absolute* justified-envy towards another student s' without sibling priority assigned to school c if (i) $g(s) = g(s')$, (ii) $c \succ_s \mu(s)$, (iii) $f(s') = \{s'\}$, and (iv) there exists a sibling $\bar{s} \in f(s) \setminus \{s\}$ such that $\mu(\bar{s}) = c$.
- A student with sibling priority s has *partial* justified-envy towards another student s' without sibling priority assigned to school c if (i) $g(s) = g(s')$, (ii) $c \succ_s \mu(s)$, and (iii) $P_\mu(s, c) > p_{s',c}$.

Note that if no students have siblings applying to the system, then partial justified-envy coincides with the standard notion of justified-envy. The concepts of absolute and partial justified-envy allow us to compare students from different

[6] A common approach used in practice is to assume that a prioritized student moves up in the order of the school until they meet their (highest ranked) siblings, displacing students with a random tie-breaker lower than the sibling who provided them with their priority.

applicant groups: students with sibling priority vs. students with no priority. Hence, it remains to describe how to compare students within the same applicant group. Among students with no priority, the schools compare their random tie-breaker and justified-envy is defined as usual. Among students with sibling priority, we define two approaches: (1) the *Dependent rule* where two students s and s' that belong to different families and they have sibling priority in the same school c, then school c compares the tie-breaker of their highest-ranked sibling already matched to c; (2) the *independent rule* where two students s and s' belong to different families and they have sibling priority in the same school c, then school c compares the tie-breaker of each competing student.

In Example 3.4 we illustrate the dependent and independent rules.

Example 3.4. Consider a single school c with capacity 3, and two families, $f = \{s_1, s_2\}$, $f' = \{s'_1, s'_2\}$, with all students applying to the same grade. Moreover, tie-breakers are such that $p_{s_1,c} > p_{s'_1,c} > p_{s'_2,c} > p_{s_2,c}$. Then, $\mu(s_1) = c$ and $\mu(s'_1) = c$. As a result, both s_2 and s'_2 get sibling priority, but there is only one seat left. If the dependent rule is in place, then $\mu(s_2) = c$ and $\mu(s'_2) = \emptyset$, since $p_{s_1,c} > p_{s'_1,c}$. On the other hand, if the independent rule is in place, $\mu(s_2) = \emptyset$ and $\mu(s'_2) = c$, since $p_{s'_2,c} > p_{s_2,c}$.

Note that the dependent and the independent rules are used in practice. On the one hand, the dependent rule is used in Chile [6], where the clearinghouse breaks ties at the family level first, and then breaks ties within each family. On the other hand, the independent rule is used in NYC to break ties among students with sibling priority.

Based on the dependent and the independent rule, we define two notions of justified-envy: (i) *dependent-justified envy* and (ii) *independent-justified envy.*

Definition 3.5 (Dependent and independent justified-envy). Consider a matching $\mu \subseteq \mathcal{E}$.

- A student with sibling priority s has *dependent* justified-envy toward another student with sibling priority s' from another family assigned to a school c if (i) $g(s) = g(s')$, (ii) $c \succ_s \mu(s)$, and (iii) $P_\mu(s,c) > P_\mu(s',c)$.
- A student with sibling priority s has *independent* justified-envy toward another student s' with sibling priority from another family assigned to a school c if (i) $g(s) = g(s')$, (ii) $c \succ_s \mu(s)$, and (iii) $p_{s,c} > p_{s',c}$.

Among students with sibling priority and from the same family, the school compares their random tie-breaker and justified-envy is defined as usual. In summary, given two students s and s' from two different families, we have:

Student s	Student s'	Justified-envy of s toward s'
No adaptive priority	No adaptive priority	Standard
Adaptive priority	No adaptive priority	Absolute or partial
No adaptive priority	Adaptive priority	Absolute or partial
Adaptive priority	Adaptive priority	Dependent or independent

Given Definitions 3.3 and 3.5, we can provide four notions of stability, as we formalize in Definition 3.6.

Definition 3.6. We say that a matching $\mu \subseteq \mathcal{E}$ is *partial-dependent stable* if it is non-wasteful, and if no student has partial and dependent justified-envy. Similarly, we define *absolute-independent stable*, *absolute-dependent stability*, *partial-independent stability* and *partial-dependent stability.*

4 EXISTENCE

As discussed in [23], stability is a desirable property since it correlates with the long-term success of the matching process. Unfortunately, as we show in Propositions 4.1 and 4.2, a stable matching under any combination of adaptive priorities, according to Definition 3.6, may not exist.

Proposition 4.1. *An absolute-(in)dependent stable matching may not exist, even if families are of size at most two.*

Proof. There are four schools, c_1, c_2, c_3, and c_4, and one single level. The schools c_1 and c_3 have one seat, and all the other schools have two seats. There are five families of students, $f_a = \{a_1, a_2\}$, $f_x = \{x\}$, $f_y = \{y\}$, $f_d = \{d_1, d_2\}$, $f_h\{h\}$. The preferences of the families (and of each student) are the following, $f_a : c_1 \succ c_2 \succ c_3 \succ c_4$; $f_x : c_2$; $f_y : c_2$; $f_d : c_1 \succ c_2 \succ c_3$; $f_h : c_4 \succ c_1$. All schools have the same tie-breaker and these are such that schools $c_1,\ c_2,\ c_3$ have the following student ordering $p_{h,c} > p_{d_1,c} > p_{x,c} > p_{y,c} > p_{d_2,c} > p_{a_1,c} > p_{a_2,c}$.

Note there is only one stable matching without sibling priority:

$$\mu = \{(a_1, c_4), (a_2, \emptyset), (x, c_2), (y, c_2), (d_1, c_1), (d_2, c_3), (h, c_4)\}.$$

Now, from this matching we can observe that student a_1 can provide sibling priority in c_4, student d_1 in c_1 and student d_2 in c_3. Then, if we sequentially apply our definition of absolute (in)dependent justified-envy, we come back to the initial matching μ. Matchings outside this cycle cannot be absolute-(in)dependent stable as that would violate Assumption 3.3: any absolute-(in)dependent stable must be derived from a stable matching in the standard sense. □

Proposition 4.2. *A partial-(in)dependent stable matching may not exist, even if families are of size at most two and there at most two grade levels.*

Proof. There are four schools, c_1, c_2, c_3, and c_4, and two levels g_1 and g_2. At level g_1, schools c_1 and c_3 have one seat, and all the other schools have two seats. At level g_2, c_1 has one seat, and all the other schools have zero seats. There are five families of students, $f_a = \{a_1, a_2\}$, $f_x = \{x\}$, $f_y = \{y\}$, $f_d = \{d_1, d_2\}$, $f_h = \{h_1, h_2\}$. All the students, except for h_2, apply to level g_1. The preferences of the students (which are the same for both levels) are the following, $f_a : c_1 \succ c_2 \succ c_3 \succ c_4$; $f_x : c_2$; $f_y : c_2$; $f_d : c_1 \succ c_2 \succ c_3$; $f_h : c_4 \succ c_1$. The random tie-breakers are the same for all schools and lead to the following student ordering $p_{h_2,c} > p_{d_1,c} > p_{x,c} > p_{y,c} > p_{d_2,c} > p_{a_1,c} > p_{h_1,c} > p_{a_2,c}$.

Note there is only one stable matching without sibling priority:

$$\mu = \{(a_1, c_4), (a_2, \emptyset), (x, c_2), (y, c_2), (d_1, c_1), (d_2, c_3), (h_1, c_4), (h_2, c_1)\}.$$

If we then try to adjust the matching according to our definitions of justified-envy, we find that we cycle. As in the proof of Proposition 4.1, any matching outside this cycle cannot be partial-(in)dependent stable. □

One important factor for the non-existence of a partial-(in)dependent stable matching is that priorities across different grade levels can go in any possible direction, i.e., there may be families where the provider of sibling priority is at a lower level and others where the provider is at a higher level. In order to mitigate this, some clearinghouses may impose additional rules, such as the one used in Chile, where there is a specified order (e.g., decreasing) in which grade levels are processed and, consequently, sibling priorities can only move according to that order (e.g., providers are in higher grades and receivers in lower ones). However, as we show in Proposition 4.3, the non-existence results hold even if we define an order in which priorities move across grade levels.

Proposition 4.3. *A partial-(in)dependent stable matching may not exist, even if families are of size at most two and there is a fixed order in which siblings provide priorities between grade levels, and there is a single tie-breaker, i.e., $p_{s,c} = p_s$ for all $c \in C$. This non-existence result also holds for absolute-(in)dependent stability.*

Proof. Let Γ be the instance provided in the proof of Proposition 4.2 with the two levels g_1 and g_2. We create another instance Γ' which is a copy of Γ where the families and the schools have different names. Moreover, the names of the two levels are switched and the agents of Γ do not rank those of Γ' and vice-versa. When we juxtapose Γ and Γ' to create a new instance Γ'', we find that for any priority ordering between grade levels g_1 and g_2 there is no partial-(in)dependent stable matching. □

4.1 Guaranteed Existence under Refined Family Preferences

An alternative explanation for the non-existence results described in the previous section is that, although students may benefit from the sibling priority, they still aim to be allocated in their most preferred school, regardless of the matching of their siblings. However, in many cases, the primary goal of the families is to get their siblings assigned to the same school. Indeed, some school districts may define as infeasible matchings where siblings are separated [9]. In other cases, the clearinghouse may explicitly elicit whether the family wants to prioritize the joint assignment of their siblings over their individual preferences. For instance, the Chilean school choice system allows families to submit a *family application*, whereby the family states that they prefer their siblings to be assigned to the same school over any assignment where this does not happen, even if the siblings end up being assigned together in a lower preferred school (see [6] for more details).

To capture these settings, we assume that families lexicographically prefer that their members are assigned to the same school over any other assignment where they are separated. For instance, if family f includes first school c and then c' in their list, then the actual preferences of each member of $s \in f$ can be written as $(c, c) \succ_s (c', c') \succ_s c \succ_s c'$, where the tuples represent that student s prefers to be assigned with at least one sibling and the non-tuples refer to individual preferences. We formalize this in Assumption 4.1.

Assumption 4.1. *Students preferences are lexicographic, so that they first prefer to be assigned with at least one of their siblings, and then to be individually assigned to the schools reported in the family list.*

Note that this assumption only restricts the space of families' preferences and, thus, does not affect the different definitions of stability under adaptive priorities stated in Definition 3.3. As we show in Theorem 4.4, if families' preferences follow Assumption 4.1 and families are of size at most two, then a partial-dependent stable matching exists.

Theorem 4.4. *A partial-dependent stable matching exists when families are of size at most two, their preferences satisfy Assumption 4.1 and there is a single tie-breaker, i.e., $p_{s,c} = p_s$ for all $c \in C$. Moreover, such a matching can be found using Algorithm 1 in $O(|\mathcal{S}|^2 \log |\mathcal{S}| + |\mathcal{S}| \cdot |C|)$.*

Proof. Since the number of agents is finite and at least one student is removed from H in each iteration, we know that Algorithm 1 finishes. Our proof consists in demonstrating the following statement by induction: At the end of every while iteration, the matching μ is such that no student in $\mathcal{S} \setminus H$ has partial-dependent justified envy.

Basis. At the end of the first iteration, one of the following three cases holds: (i) $\mu(s^\star) = \emptyset$, (ii) $\mu(s^\star) \neq \emptyset$ and $\mu(s') = \emptyset$ for all $s' \in \mathcal{S} \setminus \{s^\star\}$, or (iii) $\mu(s^\star) = \mu(s^\diamond) \neq \emptyset$. In the first case, it means that there is no school listed by $s^\star$ that has an open seat in grade $g^\star$ and, thus, $s^\star$ has no justified-envy as no school has seats open. In the second case, $s^\star$

Algorithm 1 Direct matching mechanism

Initialize: $H = \mathcal{S}$, $\mu = \emptyset$
while $H \neq \emptyset$ **do**
  Find: $s^\star = \text{argmax}_{s \in H} \{p_s\}$, $f^\star = f(s^\star) \cap H = \{s^\star, s^\diamond\}$, $g^\star = g(s^\star)$ ▷ Note: $s^\diamond = \emptyset \Leftrightarrow f^\star = \{s^\star\}$
  Initialize: $\underline{c} = \emptyset$
  for $c \in \succ_{f^\star}$ **do** ▷ In decreasing order of pref.
    if $\left|\mu^{g^\star}(c)\right| < q_c^{g^\star}$ **then**
      if $\underline{c} = \emptyset$ **then**
        Update: $\underline{c} \leftarrow c$
      if $s^\diamond = \emptyset$ **then**
        Update: $\mu \leftarrow \{(s^\star, c)\}$, $H \leftarrow H \setminus \{s^\star\}$
        break
      if $s^\diamond \neq \emptyset$ and $\left|\mu^{g(s^\diamond)}(c)\right| < q_c^{g(s^\diamond)}$ **then**
        Update: $\mu \leftarrow \{(s^\star, c), (s^\diamond, c)\}$, $H \leftarrow H \setminus f^\star$
        break
  if $s^\star \in H$ **then**
    Update: $\mu \leftarrow \{(s^\star, \underline{c})\}$, $H \leftarrow H \setminus \{s^\star\}$
return μ.

is matched to their most preferred school with seats left in $g^\star$; if $s^\diamond = \emptyset$, then $s^\star$ has no justified envy because there are no schools that $s^\star$ prefers and that have open seats. If $s^\diamond \neq \emptyset$, then it means that there is no school with seats open in $g(s^\diamond)$ and, thus, the family cannot have dependent justified envy. In the last case, family $f^\star = \{s^\star, s^\diamond\}$ is matched to their most preferred school that can accommodate both siblings at their respective grades. Note that the students in $f^\star$ cannot have justified-envy because the schools they prefer do not have enough seats to accommodate both siblings, and they prefer to be matched together over being separated.

Inductive step. Suppose that after n iterations of the while loop, no student in $\mathcal{S} \setminus H$ has partial-dependent justified envy. We need to show that this is also true at the end of the $n+1$ iteration. If $H = \emptyset$, then this holds by inductive hypothesis. Otherwise, let $s^\star$ be the student in H with the highest random tie-breaker. Then, we start searching for the acceptable schools starting from the most preferred. As soon as we find a school $\underline{c}$ with an open seat in grade $g^\star$ (if any), we record it. If none of the schools listed by family $f^\star$ have an open seat for grade $g^\star$, then $s^\star$ remains unassigned and has no justified envy towards any other matched student since they all have higher tie-breaker. Hence, we assume $\underline{c} \neq \emptyset$. If $s^\star$ has no siblings, then we simply add $(s^\star, \underline{c})$ to μ. Since all the other students previously assigned have a higher random tie-breaker than $s^\star$ and $\underline{c}$ is the most preferred school by $s^\star$ among those with seats left, then $s^\star$ cannot have justified-envy. Finally, if $s^\star$ has a sibling and school $\underline{c}$ cannot accommodate both siblings, we continue to the next preference of the family. If there is no school that can accommodate both siblings in $f^\star$, then we match $s^\star$ to $\underline{c}$ while $s^\diamond$ remains unassigned (recall that, by construction, $s^\diamond \in H$ and, thus, $p_{s^\star} > p_{s^\diamond}$). As before, $f^\star$ cannot have partial justified envy because there is no school that can accommodate both $\{s^\star, s^\diamond\}$ and all students previously assigned have a higher random tie-breaker than both siblings. Otherwise, if there is a school $\tilde{c} \preceq_{s^\star} \underline{c}$ with two open seats for the siblings in $f^\star$, then the algorithm matches both students to $\tilde{c}$. By Assumption 4.1, we know that student $s^\star$ prefers being assigned with their sibling in $\tilde{c}$ over being separated from their sibling and being assigned to $\underline{c}$, and we also know that $s^\diamond$ prefers to be matched with $s^\star$ over being separated. Finally, since there is no school they prefer that can accommodate both of them and all previously assigned students have a higher random tie-breaker, we conclude that family $f^\star$ cannot have partial-dependent justified envy.

So far, we have shown that at the end of the while loop we obtain a partial-dependent justify envy free matching μ. To show that this matching is stable, it remains to show that μ is non-wasteful. To find a contradiction, suppose it is not. Then, there exists a pair (s, c) such that $s \succ_c \emptyset$, $c \succ_s \mu(s)$, and $\left|\mu^{g(s)}(c)\right| < q_c^{g(s)}$. If s has no siblings, then we know that c had seats open in grade $g(s)$ in the iteration where s was assigned (because $\left|\mu^{g(s)}(c)\right|$ is non-decreasing in the iterations), so this leads to a contradiction as the algorithm would have assigned s to c. If s has a sibling s', there are two cases. If $\mu(s) \neq \mu(s')$, then it means that there was no school in the family's list that could accommodate both siblings, and thus they were separated. In that case, the algorithm would assign student s to school $\underline{c}$, i.e., their most preferred school with open seats. Since c had opened seats in that iteration, it means that $\underline{c} \succeq_s c \succ_s \mu(s)$, and thus s should have been assigned to $\underline{c}$. Finally, if $\mu(s) = \mu(s') \neq \emptyset$, then s prefers being assigned to c over $\mu(s)$ only if s can get assigned there with s'. However, this did not happen because, when processing student s, school c had no seats left in grade $g(s')$ (otherwise, we would have $\mu(s) = \mu(s') = c$) and, thus, given Assumption 4.1, it would not be true that s prefers to be assigned in c over their current assignment $\mu(s)$.

To conclude, note that in the worst case every student has no siblings and needs to apply to every school. Recall that a set of size $|\mathcal{S}|$ can be sorted in $O(|\mathcal{S}| \cdot \log |\mathcal{S}|)$. Steps 2-17 will be done at most $|\mathcal{S}|$ times. Step 3 takes at most $O(|\mathcal{S}| \cdot \log |\mathcal{S}|)$ and Steps 5-14 $O(|\mathcal{C}|)$. Thus, $O(|\mathcal{S}| \cdot (|\mathcal{S}| \cdot \log |\mathcal{S}| + |\mathcal{C}|)) = O(|\mathcal{S}|^2 \log |\mathcal{S}| + |\mathcal{S}| \cdot |\mathcal{C}|)$. □

Note that if there are families of size larger than two, a partial-dependent stable matching may not exist, as Example 4.5 illustrates. Nevertheless, considering that the vast majority of families that participate in these systems involve at most two siblings,[7] and that families generally prefer that their members go to the same school, this result is of high practical value. In addition, note that the proof of Theorem 4.4 is constructive, as it provides a mechanism that allows finding a stable matching under adaptive preferences. This mechanism, outlined in Algorithm 1, extends the Random Serial Dictatorship (RSD) algorithm [1] to jointly assign siblings if there is enough capacity to accommodate them. Specifically, the algorithm iterates over students in decreasing order of their random tie-breaker. If a student s has no siblings, the algorithm matches s to their most preferred school among those with seats left. If a student s has a sibling, then the algorithm stores (in $\underline{c}$) s's most preferred school with seats left in $g(s)$ and then tries to jointly assign the family $f(s)$ in order of their preferences. If a school c' has seats left in both grades, then both siblings are assigned to c'. In contrast, if no such school exists, student s is assigned to $\underline{c}$.

Proposition 4.5. *A partial-(in)dependent stable matching may not exist, even if families are of size at most three, there are at most two grades, student preferences satisfy Assumption 4.1, and there is a single tie-breaker, i.e.,* $p_{s,c} = p_s$ *for all* $c \in C$.

Proof. Consider an instance of the problem with five schools, c_1, c_2, c_3, c_4, and c_5, and two grade levels $\{g_1, g_2\}$. In g_1, c_1 and c_3 have capacity one, and all the other schools have capacity two. In g_2, c_3 has capacity one, and all the other schools have capacity zero. In addition, suppose there are five families of students, $f_a = \{a\}$, $f_b = \{b_1, b_1', b_2\}$, $f_e = \{e\}$, $f_d = \{d_1, d_1'\}$, $f_h = \{h\}$. All the students, except for b_2, apply to level g_1. The preferences of the students (which are the same for both levels) are the following, $f_a : c_1 \succ c_2$; $f_b : c_2 \succ c_3$; $f_e : c_4 \succ c_1$; $f_d : c_4 \succ c_5$; $f_h : c_3 \succ c_4$, and the random tie-breakers are such that $p_{b_2} > p_h > p_{d_1} > p_e > p_a > p_{b_1} > p_{b_1'} > p_{d_1'}$.

[7]In Chile, less than 5% of families involve more than two siblings simultaneously participating in the assignment process.

Note there is only one stable matching without sibling priority $\mu = \{(a, c_1), (b_1, c_2), (b_1', c_2), (b_2, c_3), (e, c_4), (d_1, c_4), (d_1', c_5), (h, c_3)\}$. If we then try to sequentially adjust μ following the preferences and stability assumptions of Theorem 4.4, we return to μ. Moreover, any other possible matching with sibling priority is not partial-dependent stable due to Assumption 3.3 (as argued in previous proofs). Note that this example still holds if we consider adaptive stability under partial independent justified-envy. □

As we formalize in Theorem 4.6, another case in which we can guarantee existence is when there is a single grade level. This case is also of practical relevance, since it would allow us to account for adaptive priorities in the presence of twins, and it would also capture other relevant settings such as daycare and refugee resettlement, which could be thought as having a single grade level.

Proposition 4.6. *A partial-dependent stable matching exists when there is a single grade level, families preferences satisfy Assumption 4.1 regardless of the family sizes, and there is a single tie-breaker, i.e., $p_{s,c} = p_s$ for all $c \in C$. Such a matching can be found in $O(|\mathcal{S}|^2 \log |\mathcal{S}| + |\mathcal{S}| \cdot |C|)$.*

Proof. The proof follows a similar reasoning as the one of Theorem 4.4, provided that Algorithm 1 is updated as follows. First, in Step 3, the set $f^\star$ may have more than two members. In Step 13, the algorithm may try to match as many of the siblings as possible to school c, provided that it has open seats in the corresponding grade levels. This operation can be done safely because there is a single tie-breaker, siblings have the same preferences, and we use the dependent rule to break ties among students with sibling priority. Therefore, the algorithm does not cycle, which indeed may happen in the case of families of size three on multiple levels. □

5 COMPLEXITY ANALYSIS: MAXIMUM CARDINALITY

In this section, we analyze the computational complexity of finding a partial-dependent stable matching that is of maximum cardinality, when preferences satisfy Assumption 4.1. In this setting, there may be partial-dependent stable matchings of different cardinalities, as we show in the following example.

Example 5.1. Suppose that there is one grade level, two schools c_1 and c_2 with capacities three and two, respectively, and four families $f_a = \{a\}, f_b = \{b\}, f_d = \{d\}, f_e = \{e_1, e_2\}$. The preferences of the families are $f_a : c_2 \succ c_1, f_b : c_1, f_d : c_1$, $f_e : c_1 \succ c_2$, while the random tie-breakers are as follows: for c_1 we have $p_{a,c_1} > p_{b,c_1} > p_{e_1,c_1} > p_{d,c_1} > p_{e_2,c_1}$ and for c_2 we have $p_{e_1,c_2} > p_{e_2,c_2} > p_{a,c_2}$ (f_b and f_d do not apply to c_2, so we do not need tie-breakers for them). Note that we can find two partial-dependent stable matchings of different cardinality: $\mu' = \{(a, c_1), (b, c_1), (d, c_1), (f_1, c_2), (f_2, c_2)\}$ and $\mu'' = \{(a_1, c_2), (b, c_1), (d, \emptyset), (f_1, c_1), (f_2, c_1)\}$.

As the previous example illustrates, the Rural Hospital Theorem [21, 22] does not hold in our framework. Therefore, it is essential to understand the complexity of finding the maximum cardinality partial-dependent stable matching. We now formalize the problem of finding such a matching.

Problem 5.1. *Let $\Gamma = \langle \mathcal{S}, C, \mathcal{F}, \succ, \mathbf{c}, \{p_{s,c}\}_{s\in\mathcal{S},c\in C}\rangle$ be an instance of the school choice problem with families, incomplete preference lists for the families, and let $K \in \mathbb{Z}_+$ be a non-negative integer target value. Is there a matching μ such that $|\mu| \geq K$, and μ is a partial-dependent stable matching in Γ?*

We denote this problem as Max-Card$_{FAM}$.

In Theorem 5.2, we show that Problem 5.1 is NP-complete, and we defer the proof to Appendix B. Note that this result also holds if we restrict attention to partial-independent stable matchings.

Theorem 5.2. *Problem 5.1 is NP-complete, even if there are families of size at most three and two grade levels.*

6 CONCLUSIONS

Motivated by the context of school choice with sibling priorities, we study the problem of finding a stable matching under adaptive priorities, i.e., students get prioritized if they have siblings participating in the process and who are currently assigned. We start by introducing a model of a matching market where siblings may apply together to potentially different grade levels. We argue that the standard notion of stability may not work if we allow priorities to (adapt) be a function of the matching. As a result, we define a series of assumptions on families' preferences and school priorities tailored to follow existent practices, and we introduce several novel notions of stability under adaptive preferences. Although we show that a stable matching may not exist in general, such a matching exists if families strictly prefer that their members remain together over being separated and that families are of size at most two. Moreover, we show that a stable matching also exists when there is a single grade level for any family size. Finally, we show that finding a maximum cardinality stable matching under adaptive priorities is NP-hard.

Our results show that adaptive priorities must be carefully designed to ensure the existence of a stable assignment. Moreover, several design choices have relevant implications, namely, how to break ties within and across priority groups and how families prioritize the joint assignment of their members. Hence, the insights derived from our work may help design these policies, either in the context of school choice or in other contexts, such as daycare assignments, and refugee resettlement, among others.

Our work opens several directions for future research. First, we are working on extending our existence and complexity results to the other notions of stability introduced in our work. Second, we are working on how to efficiently solve the problem of finding a stable matching under adaptive priorities using mathematical programming tools. Finally, we are collaborating with the Ministry of Education of Chile to showcase the potential benefits of considering adaptive priorities when solving the assignment of students to schools.

REFERENCES

[1] Atila Abdulkadiroğlu and Tayfun Sönmez. 1998. Random serial dictatorship and the core from random endowments in house allocation problems. *Econometrica* 66, 3 (1998), 689–701.

[2] Atila Abdulkadiroğlu and Tayfun Sönmez. 2003. School Choice: A Mechanism Design Approach. *American Economic Review* 93, 3 (may 2003), 729–747.

[3] Itai Ashlagi, Mark Braverman, and Avinatan Hassidim. 2014. Stability in Large Matching Markets with Complementarities. *Operations Research* 62, 4 (2014), 713–732.

[4] Itai Ashlagi and Peng Shi. 2014. Improving Community Cohesion in School Choice via Correlated-Lottery Implementation. *Operations Research* 62, 6 (dec 2014), 1247–1264.

[5] Federico Bobbio, Margarida Carvalho, Andrea Lodi, and Alfredo Torrico. 2022. Capacity Variation in the Many-to-one Stable Matching Problem. (2022).

[6] José Correa, Natalie Epstein, Rafael Epstein, Juan Escobar, Ignacio Rios, Nicolás Aramayo, Bastián Bahamondes, Carlos Bonet, Martin Castillo, Andres Cristi, Boris Epstein, and Felipe Subiabre. 2022. School Choice in Chile. *Operations Research* 70, 2 (mar 2022), 1066–1087.

[7] Samuel Dooley and John P Dickerson. 2020. The affiliate matching problem: On labor markets where firms are also interested in the placement of previous workers. *arXiv preprint arXiv:2009.11867* (2020).

[8] Umut Dur, Thayer Morrill, and William Phan. 2022. Family ties: School assignment with siblings. *Theoretical Economics* 17, 1 (2022), 89–120.

[9] Umut Mert Dur and Thomas Wiseman. 2019. School choice with neighbors. *Journal of Mathematical Economics* 83 (aug 2019), 101–109.

[10] David Gale and Lloyd S Shapley. 1962. College admissions and the stability of marriage. *The American Mathematical Monthly* 69, 1 (1962), 9–15.

[11] Yuichiro Kamada and Fuhito Kojima. 2015. Efficient Matching under Distributional Constraints: Theory and Applications. *The American Economic Review* 105, 1 (2015), 67–99.

[12] Bettina Klaus and Flip Klijn. 2005. Stable matchings and preferences of couples. *Journal of Economic Theory* 121, 1 (2005), 75–106.

[13] Bettina Klaus, Flip Klijn, and Jordi Massó. 2003. *Some Things Couples always wanted to know about stable matchings (but were afraid to ask).* Working Papers 78. Barcelona School of Economics. https://ideas.repec.org/p/bge/wpaper/78.html

[14] Bettina Klaus, Flip Klijn, and Toshifumi Nakamura. 2009. Corrigendum to "Stable matchings and preferences of couples"[J. Econ. Theory 121 (1)(2005) 75–106]. *Journal of Economic Theory* 144, 5 (2009), 2227–2233.
[15] Marina Knittel, Samuel Dooley, and John Dickerson. 2022. The Dichotomous Affiliate Stable Matching Problem: Approval-Based Matching with Applicant-Employer Relations. In *Proceedings of the Thirty-First International Joint Conference on Artificial Intelligence, IJCAI-22*, Lud De Raedt (Ed.). International Joint Conferences on Artificial Intelligence Organization, 356–362. https://doi.org/10.24963/ijcai.2022/51 Main Track.
[16] Fuhito Kojima, Parag A Pathak, and Alvin E Roth. 2013. Matching with couples: Stability and incentives in large markets. *The Quarterly Journal of Economics* 128, 4 (2013), 1585–1632.
[17] David F Manlove, Robert W Irving, Kazuo Iwama, Shuichi Miyazaki, and Yasufumi Morita. 2002. Hard variants of stable marriage. *Theoretical Computer Science* 276, 1-2 (2002), 261–279.
[18] Hai Nguyen, Thành Nguyen, and Alexander Teytelboym. 2021. Stability in Matching Markets with Complex Constraints. *Management Science* 67, 12 (2021), 7438–7454.
[19] Thành Nguyen and Rakesh Vohra. 2018. Near-feasible stable matchings with couples. *American Economic Review* 108, 11 (2018), 3154–69.
[20] Thành Nguyen and Rakesh Vohra. 2019. Stable Matching with Proportionality Constraints. *Operations Research* 67, 6 (2019), 1503–1519.
[21] Alvin E Roth. 1984. The evolution of the labor market for medical interns and residents: a case study in game theory. *Journal of political Economy* 92, 6 (1984), 991–1016.
[22] Alvin E Roth. 1986. On the allocation of residents to rural hospitals: a general property of two-sided matching markets. *Econometrica: Journal of the Econometric Society* (1986), 425–427.
[23] Alvin E. Roth. 2002. The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics. *Econometrica* 70, 4 (July 2002), 1341–1378.

A EXTRA DISCUSSION ON HOW TO PROCESS GRADE LEVELS AND OTHERS

As proposed in [6], one option to handle adaptive priorities is to define an order in which grades are processed and sequentially solve the assignment of each grade level using the student-optimal variant of DA. More specifically, the algorithm in [6] starts processing the highest grade (i.e., 12th grade). Then, before moving to the next grade, the sibling priorities are updated, considering the assignment of the grade levels already processed. After processing the final grade level (i.e., Pre-K), this procedure finishes. Notice that this heuristic obtains a stable assignment if the preferences of families satisfy *higher-first*, i.e., each family prioritizes the assignment of their oldest member (see Proposition 2 in [6]). However, this is not the case if some families' preferences do not satisfy this condition. In addition, as Example A.1 illustrates, the order in which grades are processed matters.

Example A.1. Consider an instance with two grades $g_1 < g_2$, two schools c_1 and c_2 with one seat in each grade, one family $f = \{f_1, f_2\}$, and two additional students, a_1 and a_2. Students f_1 and a_1 apply to grade g_1, and f_2 and a_2 apply to grade g_2. Finally, the preferences and priorities are:

$$\begin{aligned}
&(c_2, c_1) \succ_f (c_1, c_1) \succ_f (c_2, c_2) \succ_f (c_1, c_2) \\
&c_2 \succ_{a_1} c_1 \\
&c_1 \succ_{a_2} c_2 \\
&p_{a_1,c_1} > p_{f_1,c_1} \text{ and } p_{a_2,c_1} > p_{f_2,c_1} \\
&p_{a_1,c_2} > p_{f_1,c_2} \text{ and } p_{a_2,c_2} > p_{f_2,c_2}.
\end{aligned} \tag{1}$$

Since the preferences $\succ_f$ are *responsive*, we can easily derive the related individual preferences $\succ_{f_1}$ and $\succ_{f_2}$, which are $c_2 \succ_{f_1} c_1$ and $c_1 \succ_{f_2} c_2$ [12, 13]. We observe that, if grades are processed in decreasing order (as in Chile), we obtain the matching $\mu = \{(f_1, c_2), (a_1, c_1), (f_2, c_2), (a_2, c_1)\}$. In contrast, if we process grades in increasing order, we obtain the matching $\mu' = \{(f_1, c_1), (a_1, c_2), (f_2, c_1), (a_2, c_2)\}$. □

B MISSING PROOFS IN SECTION 5

Our reduction is done from the problem of finding a maximum cardinality stable matching in a market where schools are partitioned in sets and each set of schools receives some extra seats that should be allocated optimally.

Problem B.1. *Let $\Gamma = \langle \mathcal{S}, C, \mathcal{F}, \succ, \mathbf{c}, \{p_{s,c}\}_{s\in\mathcal{S},c\in C}\rangle$ be an instance of the school choice problem with families, incomplete preference lists for the families, a partition $\mathcal{P} = \{C_1, \ldots, C_q\}$ of C, budget for each part $\{B_k \in \mathbb{Z}_+ : k \in [q]\}$, and let $K \in \mathbb{Z}_+$ be a non-negative integer target value. Is there a non-negative allocation vector $\mathbf{t} \in \mathbb{Z}_+^C$ and a matching $\mu_\mathbf{t}$ such that $|\mu_\mathbf{t}| \geq K$, where $\mathbf{t}$ is such that $\sum_{j\in C_k} t_j \leq B_k$ for each $k \in [q]$ and μ is a stable matching in the expanded instance Γ_t?*

We denote this problem as Max-Card$^{\text{SUB}}_{\text{EXP}}$.

Proof of Theorem 5.2. We provide a reduction from Max-Card$^{\text{SUB}}_{\text{EXP}}$ [5], the problem of finding a maximum cardinality stable matching in a instance where preferences may be incomplete, schools are partitioned into subsets and every such set has a budget to expand schools' capacities. From the proof of Theorem 5.2 in [5], we deduce the following assumptions on the generic instance of Max-Card$^{\text{SUB}}_{\text{EXP}}$.

- The partition of schools $\mathcal{P} = \{C_1, \ldots, C_p\}$ is made of subsets of size at most two and for each such C_k there is an extra budget $B_k \leq 1$ for $k \leq l$.
- Every school that is partitioned as a singleton has capacity one and an extra capacity zero. We denote by $C^\star$ the set of these schools.
- Every school that is partitioned in a subset of cardinality two (C_k), has capacity zero; the budget of extra capacities allocated to such pair of schools is one, $B_k = 1$. We denote by $C^{\star\star}$ the set of schools with capacity zero; for simplicity we assume that C_i with $i \leq t$ we have the schools with capacity zero. Every school with capacity zero has a preference list of length one, i.e., it ranks only one student; moreover, we make the crucial assumption that one of the two students ranked by the pair of schools is ranked only once by a pair of schools.[8]

The objective in instance Γ is to find a stable matching of cardinality at least K. Given an instance Γ of Problem 5.1, we build an instance Γ' of Max-Card$_{\text{FAM}}$ with two grades, g_1, g_2, and families of size at most three. First of all, we create a copy of $\mathcal{S}$ in Γ'. For every school c in $C^\star$ we create a copy of c in Γ' with the same preference list and the same capacity at level g_1. For a school c in $C^{\star\star}$, let $C_r = \{c'_r, c''_r\}$ be the subset of the partition $\mathcal{P}$ to which c belongs. Assume the preference lists of c'_r, c''_r in Γ are $c'_r : s'_r$ and $c''_r : s''_r$, respectively. In Γ' we create two new schools $\bar{c}_r$ and c^+_r (note that we make no copies of c'_r and c''_r) and three new students y'_r, y''_r and w_r. In Γ', students s'_r, s''_r, w_r and y'_r apply to level g_1, while y''_r applies to level g_2. All the other students in Γ that are not ranked by a school in $C^{\star\star}$, apply in Γ' to grade g_1. On the other hand, school c^+_r has only one capacity at level g_1 and one capacity at level g_2; school $\bar{c}_r$ has two capacities at level g_1 and no seats at level g_2. We assume s''_r is the student that is only ranked once by a pair of schools in Γ, and we create the family $f_r = \{s''_r, y'_r, y''_r\}$ in Γ'. The preferences of these agents in Γ' are as follows.

- $\bar{c}_r : y'_r \succ s'_r \succ w_r \succ s''_r \succ y''_r$.
- $c^+_r : w_r \succ y'_r \succ y''_r \succ s''_r$.
- $w_r : \bar{c}_r \succ c^+_r$.
- In the original preference list of s''_r we substitute c'_r with $c^+_r \succ \bar{c}_r$. Therefore, the preference list of f_r is the same as s''_r. All the schools ranked by f_r that are not $\{c^+_r, \bar{c}_r\}$, they rank y'_r and y''_r last.

[8] In [5],the authors prove that Max-Card$^{\text{SUB}}_{\text{EXP}}$ is NP-hard by proving a reduction from the problem of finding the maximum cadinality stable matching with ties and incomplete lists [17]. The assumption that we made (the fact that each school in a pair lists only one student), follows from the fact that in the proof of Theorem 2 in [17], the preference list of $x_{i,r}$ is only made by a tie of length two. The second assumption (the fact that one of the two students ranked by the pair is only listed by one pair) follows from the fact that $w_{i,r}$ is only ranked by $x_{i,r}$ in the proof of Theorem 2 in [17].

- In the original preference list of s'_r we substitute c''_r with $\bar{c}_r$.

From the assumption that s''_r is ranked only once by a pair in Γ, it follows that s''_r has only two siblings, which are exactly y'_r and y''_r. Note that c^+_r and y''_r will always be matched together. Let l be the number of paired sets from $\mathcal{P}$ in Γ (i.e., half the number of schools in $C^{\star\star}$).

Let M be a stable matching in Γ of cardinality at least K. Our goal is to find a corresponding partial-dependent stable matching M' in Γ' that has a cardinality at least $K + 3 \cdot l$. Let (c, s) be a pair in M. If c is in $C^\star$, then we match (c, s) in M'. Otherwise, c is part of a pair $C_r = \{c'_r, c''_r\}$. If $c = c'_r$, then we match $(\bar{c}_r, s'_r)$, $(\bar{c}_r, w_r)$, (c^+_r, y''_r) and y'_r with a school ranked at least as c^+_r. On the other hand, if $c = c''_r$, then we match $(\bar{c}_r, s''_r)$, $(\bar{c}_r, y'_r)$, (c^+_r, w_r), and (c^+_r, y''_r). Finally, it may happen that some schools in $C^{\star\star}$ may be under-subscribed in Γ; in this case, we match $(\bar{c}_r, w_r)$, (c^+_r, y''_r), and y'_r with a school ranked at least as c^+_r. We need to prove that the matching M' that we just built is partial-dependent stable in Γ'. First, the pairs involving schools in $C^\star$ directly inherit the stability from M. Note that the set of students $\{y''_r, y'_r\}$ will always be matched: As mentioned earlier, y''_r will always be matched to c^+_r, while y'_r will be matched to a school that is ranked at least as $\bar{c}_r$. Indeed, y'_r can be matched to c^+_r, $\bar{c}_r$ or a school more preferred than these, which is under-subscribed in matching Γ. Similarly, also student w_r is always matched to the set of schools $\{\bar{c}_r, c^+_r\}$. Clearly, y''_r cannot have partial-dependent justified envy because it is assigned to the only acceptable school with a spot in g_2. If s'_r is matched to $\bar{c}_r$, then s''_r cannot have partial-dependent justified envy because it is matched or to a better school than $\bar{c}_r$, or because it does not receive priority from y'_r (which, in this case, is matched to c^+_r - recall that $y'_r \succ_{c^+_r} s''_r$). If, instead, s''_r is matched to $\bar{c}_r$, then s'_r, w_r cannot have partial-dependent justified envy because s''_r uses the priority obtained by y'_r. Also y'_r cannot have partial-dependent justified envy: c^+_r is matched to its preferred student (w_r) and all the schools more preferred than $\bar{c}_r$, prefer s''_r to y'_r. Finally, note that if neither s'_r nor s''_r are matched to $\bar{c}_r$, then none of the students in the pairs $(\bar{c}_r, w_r)$, (c^+_r, y''_r), and $(y'_r, \tilde{c})$ (where $\tilde{c}$ is a school ranked at least as c^+_r) may have partial-dependent justified envy. Therefore, we have built a partial-dependent stable matching in Γ. Note that for every pair of schools in $C^{\star\star}$, we have introduced three new students that are always matched. Hence, we obtained a matching M' of cardinality $K + 3 \cdot l$.

Let M' be a partial-dependent stable matching in Γ' of cardinality K'. Given $r \leq l$, w_r will always be matched to a school in $\{\bar{c}_r, c^+_r\}$, y''_r will always be matched to c^+_r, and y'_r to a school ranked at least as $\bar{c}_r$. Therefore, $3 \cdot l$ students will always be matched in M'. Our goal is to find a corresponding stable matching M in Γ of cardinality at least $K = K' - 3 \cdot l$. For every student s matched to a school c in $C^\star$ in Γ', we match (s, c) in Γ. On the other hand, if a student s is matched to a school $\bar{c}_r$ for a certain $r \leq l$, then, if $s = s'_r$, in Γ we match (s'_r, c'_r) by allocating one extra capacity to c'_r; otherwise, if $s = s''_r$, in Γ we match (s''_r, c''_r) by allocating one extra capacity to c''_r. If some $\bar{c}_r$ does not match any student of the form s'_r, s''_r, then both students are matched to a school they rank better; hence, in Γ, we can allocate the extra spot arbitrarily to c'_r and c''_r. We now prove the following statement: If some $\bar{c}_r$ does not match any student of the form s'_r, s''_r, then both students are matched to a school they rank better. If the previous statement is true, then the matching M in Γ is stable since it mimics the stability of M' without needing the students of the form $\{y'_r, y''_r, w_r\}_{r\leq l}$ which only make sure we can do the other inclusion; hence, we can obtain M as a restriction of matching M' in Γ' without considering the students $\{y'_r, y''_r, w_r\}_{r\leq l}$. To prove the statement, let us assume that there is $r \leq l$ for which s'_r and s''_r are not matched to $\bar{c}_r$ and at least one of the two students is matched to a school ranked worst. Assume s'_r is matched to $\tilde{c}$ such that $\bar{c}_r \succ_{s'_r} \tilde{c}$. Then, $(s'_r, \bar{c}_r)$ is a blocking pair in Γ' since $\bar{c}_r$ has capacity two and s'_r is ranked second by $\bar{c}_r$. Therefore, s'_r is matched to a school more preferred than $\bar{c}_r$. Now let us assume that s''_r is matched to $\tilde{c}$ such that $\bar{c}_r \succ_{s''_r} \tilde{c}$. Then, $(s''_r, \bar{c}_r)$ is a blocking pair in Γ' for one of the following two reasons: 1) y'_r is matched to $\bar{c}_r$ and s''_r can use sibling priority, 2) there is one empty capacity in $\bar{c}_r$ therefore the condition of non-wastefulness is not met. Finally, note that the students matched in Γ' that are not of the form $\{y'_r, y''_r, w_r\}_{r \leq l}$ are exactly K, which is the number of students that we matched in M.

Note that the proof holds even if we set $K = n$, i.e., we are looking for a perfect stable matching.

To build our new instance, we have introduced $3 \cdot l$ new students, and we have substituted l schools with $2 \cdot l$ new schools. Since $l \leq |\mathcal{S}|$, the reduction that we built is polynomial.

□